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Special Lagrangian submanifolds and Lagrangian mean curvature flows with generalized perpendicular symmetries

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Abstract. This is a survey on the author's recent work [22] and a pre-report on [23]. We show a method of constructing an invariant Lagrangian mean curvature flow in a Calabi-Yau manifold with the use of generalized perpendicular symmetries. We also show a way to construct special Lagrangian submanifolds as a special case of our method. We use moment maps of the actions of Lie groups, which are not necessarily abelian. By our method, we construct a non-trivial examples of special Lagrangian submanifolds in the cotangent bundle of the n-sphere T^*S^n and a self-similar solution of a Lagrangian mean curvature flow in $\mathbb{C}^n$.

1. Introduction

Calabi-Yau manifolds have received much attentions as a model of the string theory in physics. The mirror symmetry is one of remarkable properties of Calabi-Yau manifolds which is useful to understand themselves. The Strominger–Yau–Zaslow conjecture [25] explains that a Calabi-Yau manifold and its mirror are both interpreted as a special Lagrangian torus fibration with the same base manifold respectively. This conjecture indicates that it is important to see Calabi-Yau manifolds from the perspective of special Lagrangian submanifolds.

Another context which shows importance of special Lagrangian submanifolds is in calibrated geometry. Calibrated submanifolds were introduced in [5] as classes of submanifolds in Riemannian manifolds. For a certain holomorphic volume form Ω called a Calabi-Yau structure in a Calabi-Yau manifold, a submanifold calibrated by the real part of $e^{\sqrt{-1}\theta}\Omega$ ($\theta \in \mathbb{R}$) is called a special Lagrangian submanifold. It is known that a calibrated submanifold is volume minimizing in its homological class and so is a special Lagrangian submanifold.

Joyce gave various examples of special Lagrangian submanifolds in $\mathbb{C}^n$ in a series

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of his papers [13]–[17]. One of typical methods of constructing special Lagrangian submanifolds is called the moment map technique which was introduced by Joyce in [16]. By this method, Joyce constructed special Lagrangian submanifolds in $\mathbb{C}^n (\cong \mathrm{T}^*\mathbb{R}^n)$ which is invariant under a subgroup of $SU(n)$. By the moment technique, Anciaux [2], Ionel–Min-Oo [10], Hashimoto–Sakai [7], and Hashimoto–Mashimo [6] constructed special Lagrangian submanifolds in the cotangent bundle of the n-sphere T^*S^n, and Arai–Baba [3] in the cotangent bundle of the complex projective space $\mathrm{T}^*\mathbb{C}P^n$. Each ambient space is a non-flat Calabi-Yau manifold introduced by Stenzel [24] and all of these examples are cohomogeneity one.

Another method of constructing special Lagrangian submanifolds is called the bundle technique which was introduced by Harvey–Lawson [5]. By the bundle technique Karigiannis–Min-Oo [18] constructed special Lagrangian submanifolds in T^*S^n, and Ionel–Ivey [9] in $\mathrm{T}^*\mathbb{C}P^n$.

Aside from these two methods, Joyce [16] showed a way to construct special Lagrangian submanifolds in $\mathbb{C}^n$ by using actions of abelian subgroups of $SU(n)$ which act perpendicularly to another given special Lagrangian submanifold. The method of our paper of constructing special Lagrangian submanifolds is a generalization of this method.

Lagrangian mean curvature flows give us another method of constructing special Lagrangian submanifolds from a different point of view. A mean curvature flow gives a standard deformation of an immersion into a Riemannian manifold. It is known that the deformation by a mean curvature flow reduces the volume of an immersion most efficiently. So it is considered as a fundamental tool for finding minimal submanifolds. In particular, when an ambient space is a Kähler-Einstein manifold, it is known that a mean curvature flow of a Lagrangian immersion preserves its Lagrangian condition. This indicates that a Lagrangian mean curvature flow gives a method of finding a minimal Lagrangian submanifold (especially a special Lagrangian submanifold in a Calabi-Yau manifold).

One of main problems on mean curvature flows is when they converge to minimal submanifolds, especially when they converge to special Lagrangian submanifolds in a Calabi-Yau manifold. In [27] Thomas–Yau defined a notion of stability for Lagrangian submanifolds in a Calabi-Yau manifold. They conjectured that if a Lagrangian submanifold is stable, then its Lagrangian mean curvature flow exists for all time and converges to a special Lagrangian submanifold. Recently this conjecture has been reformulated by Joyce in [12].

Another main problem lies in singularities of the flow. Huisken [8] showed that if a mean curvature flow has a type I singularity, then a self-similar solution of a mean curvature flow appears as the blow-up limit of a rescaled flow. It is also known that if a mean curvature flow has another type of singularities, then a certain blow-up limit is a translating soliton. Thus, they are important classes of mean curvature flows as local models of singularities.

Since a mean curvature flow is written by a partial differential equation, it is generally difficult to solve them and construct examples. Several examples of self-similar solutions or translating solitons of Lagrangian mean curvature flows are known as follows. Anciaux [1] and Lee–Wang [20, 21] constructed self-similar solutions. Joyce–Lee–Tsui [11] also constructed self-similar solutions and translating solitons. Castro–Lerma [4] constructed translating solitons in $\mathbb{C}^2$. In [28], Yamamoto indicated that some of them can be explained in terms of moment maps and toric symmetries. He showed a way to construct a generalized mean curvature flow in a toric almost Calabi-Yau manifold. In 2019, Konno [19] generalized this result to a way using moment maps and perpendicular symmetries of abelian Lie groups. By this method, he constructed Lagrangian mean curvature flows in ALE spaces which are hyperKähler manifolds. This is considered as a first example of a construction of a Lagrangian mean curvature flow in a non-flat Calabi-Yau manifold.

The overview of Konno's method is as follows. Let M be a Calabi-Yau manifold, L a special Lagrangian submanifold of M, H an abelian Lie group which has a moment map and acts on M perpendicularly to L. Then there exists another H-invariant Lagrangian submanifold, and its H-invariant Lagrangian mean curvature flow can be described under some assumptions. Konno constructed some examples of self-similar solutions and translating solitons in $\mathbb{C}^n$ by this method. Moreover, he showed that these self-similar solutions in $\mathbb{C}^n$ can be observed at singularities of Lagrangian mean curvature flows which he constructed in ALE spaces. Considering special Lagrangian submanifolds as stationary solutions of Lagrangian mean curvature flows, this method can be interpreted as a generalization of Joyce's result [16] which showed a way to construct special Lagrangian manifolds in $\mathbb{C}^n$ by perpendicular symmetries of abelian Lie subgroups of $SU(n)$ as mentioned above.

In the author's previous paper [22], by generalizing Joyce and Konno's method, he showed a way to construct special Lagrangian submanifold in Calabi-Yau manifolds by generalized perpendicular symmetries of Lie groups which is not necessarily abelian and he constructed non-trivial examples in the cotangent bundles of spheres T^*S^n equipped with the Stenzel metric. Recently in [23], the author also generalized the Konno's result in two points for constructing Lagrangian mean curvature flows. First, we do not assume commutativity of Lie groups. Second, we generalize the condition that Lie groups act on M perpendicularly to L. We show an example of self-similar solution in $\mathbb{C}^n$ by our method. This is a generalization of examples by Lee–Wang [20] and Konno [19].

Moreover, we generalize this method to one of constructing an H-invariant mean curvature flow in a general Riemannian manifold. We also show a way to reduce the partial differential equation of a mean curvature flow in a Riemannian manifold to an ordinary differential equation with the use of Lie group actions.

The overview of this survey article is as follows. In Section 3, we show a way to construct an H-invariant mean curvature flow in a Riemannian manifold. In Section 4, we show that the mean curvature vector field of an H-invariant Lagrangian immersion into a Calabi-Yau manifold is completely expressed in terms of Lie group actions under some assumptions. In Section 5, we show a method of constructing Lagrangian mean curvature flows in Calabi-Yau manifolds by generalized perpendicular symmetries, based on the results in Section 4. Finally, we show some examples of special Lagrangian submanifolds in T^*S^n and of a self-similar solution in $\mathbb{C}^n$ by our method in Section 6 and Section 7 respectively.

2. Preliminaries

In this section, we review some fundamental facts on Calabi-Yau manifolds, their special Lagrangian submanifolds, Lie group actions, and moment maps.

2.1. Lagrangian submanifolds in a Calabi-Yau manifold

We begin with the definition of Lagrangian submanifolds in symplectic manifolds.

Let (M, ω) be a symplectic manifold of (real) dimension $2n$. A submanifold L of (M, ω) is called *isotropic* if $\omega|_L \equiv 0$. If an isotropic submanifold L is of half-dimension of $\dim M$, it is called a *Lagrangian submanifold*.

Next we see the definition of special Lagrangian submanifolds. It is a particular submanifold of a Calabi-Yau manifold which is defined as follows.

Definition 2.1. A *Calabi-Yau manifold* is a quadruple (M, I, ω, Ω) such that (M, I) is a complex manifold equipped with a Kähler form ω and a holomorphic volume form Ω which satisfy the following relation:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

Definition 2.2. If L is an oriented Lagrangian submanifold of a Calabi-Yau manifold (M, I, ω, Ω), there exists a function $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$, which is called the

Lagrangian angle satisfying

$$\iota^*\Omega = e^{\sqrt{-1}\theta}\mathrm{vol}_{\iota^* g}.$$

Here g is the Kähler metric, $\iota : L \to M$ is an embedding, and $\mathrm{vol}_{\iota^* g}$ is the volume form on L with respect to the induced metric $\iota^* g$.

Even if L is not orientable, we can locally define the Lagrangian angle with the formula above. With the use of the Lagrangian angle θ of a Lagrangian submanifold L, the mean curvature vector field $\mathcal{H}$ of L is expressed as the following proposition.

PROPOSITION 2.3. *It holds that*

$$\mathcal{H}(p) = I_{\iota(p)}(\iota_{*p}(\nabla_{\iota^* g}\theta)_p) \in \mathrm{T}^{\perp}_{\iota(p)}\iota(L) \quad (p \in L),$$

where $\nabla_{\iota^* g}\theta$ is the gradient of the function θ with respect to the induced metric $\iota^* g$.

The definition of a special Lagrangian submanifold is given by the following.

DEFINITION 2.4. Let (M, I, ω, Ω) be a Calabi-Yau manifold. A *special Lagrangian submanifold* of (M, I, ω, Ω) is a Lagrangian submanifold such that its Lagrangian angle is constant $\theta \equiv \theta_0$. The constant θ_0 is called the *phase* of the special Lagrangian submanifold.

From the formula of the mean curvature vector in Proposition 2.3, we can see that a special Lagrangian submanifold is a minimal submanifold. More strongly, it is known that a special Lagrangian submanifold is homologically volume minimizing.

2.2. Lie group actions and moment maps

In this subsection we review the fundamental notions of Lie group actions and moment maps.

Let M be a manifold, H a Lie group which acts on M. We denote the translation of $h \in H$ by $L_h : M \to M;\ p \mapsto L_h(p) = hp$. For each $p \in M$, the orbit and the isotropy subgroup at p are denoted by $H \cdot p$ and H_p respectively.

Letting $\mathfrak{h}$ denote the Lie algebra of H, any $\xi \in \mathfrak{h}$ induces a fundamental vector field $\xi^{\#}$ on M, defined as follows.

$$\xi^{\#}_p = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)p \quad (p \in M),$$

where $\exp(t\xi)$ denotes the 1-parameter subgroup of H associated to ξ.

H acts on its Lie coalgebra $\mathfrak{h}^*$ by the *coadjoint action* $\mathrm{Ad}_h^* : \mathfrak{h}^* \to \mathfrak{h}^*$, where $h \in H$, and for $c \in \mathfrak{h}^*$, $\mathrm{Ad}_h^* c$ is defined by the following.

$$\langle \mathrm{Ad}_h^* c, \xi \rangle = \langle c, \mathrm{Ad}_{h^{-1}} \xi \rangle \quad (\xi \in \mathfrak{h}).$$

Here $\langle \cdot, \cdot \rangle$ is the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$. We call

$$Z(\mathfrak{h}^*) = \{c \in \mathfrak{h}^* \mid \mathrm{Ad}_h^* c = c, \ \forall h \in H\}$$

the *center* of $\mathfrak{h}^*$. If H is abelian, then $Z(\mathfrak{h}^*) = \mathfrak{h}^*$ holds.

Next, let (M, ω) be a symplectic manifold and define a moment map $\mu : M \to \mathfrak{h}^*$ as follows.

Definition 2.5. Let H be a Lie group acting on a symplectic manifold (M, ω). A *moment map* $\mu : M \to \mathfrak{h}^*$ is an H-equivariant map that satisfies

$$-\mathfrak{i}(\xi^{\#})\omega = d\langle \mu(\cdot), \xi \rangle \quad (\xi \in \mathfrak{h}),$$

where $\mathfrak{i}$ is the interior product.

If (M, ω, H) has a moment map, the H-action is called *Hamiltonian.* A Hamiltonian action preserves ω. For each $c \in \mathfrak{h}^*$ and $p \in \mu^{-1}(c)$, the orbit $H \cdot p$ is isotropic if and only if $c \in Z(\mathfrak{h}^*)$.

3. Constructions of mean curvature flows by symmetries

In this section, we firstly study some fundamental facts about immersions in relation to actions of Lie groups. We secondly define a notion which expresses a sort of symmetries of mean curvature vector fields of an immersion, using actions of Lie groups. We thirdly show a way to construct mean curvature flows by such symmetries of Lie groups.

Before that, we define some notations which we use throughout the remainder of this paper. Let M be a manifold, H a Lie group which acts on M, K a closed subgroup of H. For any submanifold N of M, we define $N^K := \{p \in N \mid H_p = K\}$. In particular, the subset M^K is defined by $M^K = \{p \in M \mid H_p = K\}$. For each submanifold V of M such that $V \subset M^K$, we define a map $\phi_V : (H/K) \times V \to M$ by $(hK, p) \mapsto hp$. For the sake of the condition $V \subset M^K$, this map is well-defined.

The differential map $(\phi_V)_*$ defines a linear map $(\phi_V)_{*(hK,p)} : \mathrm{T}_{hK}(H/K) \times$

$\mathrm{T}_pV \to \mathrm{T}_{hp}M$ for each $(hK, p) \in (H/K) \times V$. It holds that

$$\mathrm{T}_{hK}(H/K) = \left\{ \left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)K \;\middle|\; \xi \in \mathfrak{h} \right\}. \tag{1}$$

In fact, for any $g \in H$, define a map τ_g by

$$\tau_g : H/K \to H/K; hK \mapsto ghK.$$

The map τ_g is a diffeomorphism. We have $\mathrm{T}_K(H/K) = \left\{ \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)K \mid \xi \in \mathfrak{h} \right\}$. We also have

$$(\tau_h)_{*K} \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)K = \left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)K.$$

Since the linear map $(\tau_h)_{*K} : \mathrm{T}_K(H/K) \to \mathrm{T}_{hK}(H/K)$ is an isomorphism, the claim (1) holds.

We define two maps, a projection $\pi : H \to H/K$ by $h \mapsto hK$ and a diffeomorphism $j : H/K \to H \cdot p$ by $hK \to hp$. Then we obtain their differential maps, a submersion $(d\pi)_K : \mathfrak{h} \to \mathrm{T}_K(H/K)$ defined by $\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)K$ and a linear isomorphism $(dj)_K : \mathrm{T}_K(H/K) \to \mathrm{T}_p(H \cdot p)$ defined by $\left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)K \to \xi_p^{\#}$.

The next proposition holds whether the map ϕ_V is an immersion or not.

Proposition 3.1 ([22]). *Let M be a manifold, H a Lie group which acts on M, K a closed subgroup of H, and V a submanifold of M such that $V \subset M^K$. For any $(hK, p) \in (H/K) \times V$, $\xi \in \mathfrak{h}$, and $v \in \mathrm{T}_pV$, it holds that*

$$(\phi_V)_{*(hK,p)} \left(\left.\frac{d}{dt}\right|_{t=0} h\exp(t\xi)K, v \right) = (L_h)_{*p}(\xi_p^{\#} + v).$$

We show the condition that the map ϕ_V becomes an immersion.

Proposition 3.2 ([22]). *Let M be a manifold, H a Lie group which acts on M, K a closed subgroup of H, and V a submanifold of M such that $V \subset M^K$. The map ϕ_V is an immersion if and only if it holds that*

$$\xi_p^{\#} \notin \mathrm{T}_pV \setminus \{0\} \quad (p \in V, \xi \in \mathfrak{h}).$$

Next, we define some notions which are related to some symmetries of mean curvature vector fields.

Definition 3.3. Let $\phi : \Sigma \to M$ be an immersion from a k-dimensional manifold Σ to a manifold M. A smooth map $f : \Sigma \times [0,T) \to M; (p,t) \mapsto f_t(p)$ such that $f_0 = \phi$ is called a *deformation* of ϕ if for each $t \in [0,T)$, the map $f_t(\cdot) : \Sigma \to M$ is an immersion. In particular, when an immersion ϕ equals the inclusion map $\iota : \Sigma \to M$, we also call f a deformation of Σ.

If there exists a deformation f of an immersion $\phi : \Sigma \to M$, the set $f_t(\Sigma)$ is a k-dimensional immersed submanifold for each $t \in [0,T)$.

Definition 3.4. Let (M,g) be a Riemannian manifold, Σ a manifold, $\phi : \Sigma \to M$ an immersion. A *mean curvature flow* $F = (F_t)_{t\in[0,T)}$ of ϕ is a deformation of ϕ which is a smooth solution of the following partial differential equation:

$$\frac{\partial}{\partial t}F(p,t) = \mathcal{H}^t(p),$$

where $\mathcal{H}^t$ denotes the mean curvature vector field of the immersion F_t for each $t \in [0,T)$.

It is known that in a Kähler-Einstein manifold, any mean curvature flow of a Lagrangian immersion ϕ preserves its Lagrangian condition, i.e., if there exists a mean curvature flow $(F_t)_{t\in[0,t)}$ of a Lagrangian immersion $\phi : L \to M$, then the immersed submanifold $F_t(L)$ is also Lagrangian for each time $t \in [0,T)$.

When M is the Euclidean space, there are some important classes of solutions of mean curvature flows.

Definition 3.5. Let $(\mathbb{R}^n, g)$ be the Euclidean space equipped with the standard Riemannian metric g, Σ a manifold, $\phi : \Sigma \to \mathbb{R}^n$ an immersion, $\mathcal{H}$ a mean curvature vector field of ϕ. If there exists a constant $\lambda \in \mathbb{R}$ and it holds that

$$\mathcal{H}(p) = \lambda\phi^\perp(p) \quad (p \in \Sigma), \tag{2}$$

the solution of the mean curvature flow of the immersion ϕ is called a *self-similar solution*, where $\phi^\perp(p)$ denotes the normal part of the position vector $\phi(p) \in \mathbb{R}^n$. In particular, such a solution is called a *self-shrinker* if $\lambda < 0$ and a *self-expander* if $\lambda > 0$.

Definition 3.6. Let $(\mathbb{R}^n, g)$ be the Euclidean space equipped with the standard Riemannian metric g, Σ a manifold, $\phi : \Sigma \to \mathbb{R}^n$ an immersion, $\mathcal{H}$ a mean curvature vector field of ϕ. If there exists a constant vector $v \in \mathbb{R}^n$ and it holds that

$$\mathcal{H}(p) = v^{\perp}(p), \tag{3}$$

the solution of the mean curvature flow of the immersion ϕ is called a *translating soliton*, where $v^{\perp}(p)$ denotes the normal part of the vector v at $\phi(p) \in \mathbb{R}^n$.

Under some assumptions, we can construct another deformation F from a deformation f with the use of Lie group actions.

DEFINITION 3.7. Let M be a manifold, H a Lie group which acts on M, K a closed subgroup of H, V_0 a submanifold of M such that $V_0 \subset M^K$, $f : V_0 \times [0,T) \to M^K$ a deformation of V_0. Suppose that for each $t \in [0,T)$ and an immersed submanifold $V_t := f_t(V_0)$, the map $\phi_{V_t} : (H/K) \times V_t \to M$ is also an immersion. Then we can define a deformation F of ϕ_{V_0} by

$$F : (H/K) \times V_0 \times [0,T) \to M; \quad (hK, p, t) \mapsto hf_t(p) =: F_t(hK, p).$$

We call the deformation F the *expansion* of the deformation f by the H-action.

We express some symmetries on mean curvature vector fields of an immersion ϕ_V as follows.

DEFINITION 3.8. Let (M, g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, V a submanifold of M such that $V \subset M^K$. We say that V has the *property* $(*)$ with respect to the H-action if the map ϕ_V is an immersion and it holds that

$$\mathcal{H}(hK, p) = (L_h)_{*p}\mathcal{H}(K, p) \quad (h \in H, p \in V), \tag{$*$}$$

where $\mathcal{H}$ is the mean curvature vector field of the immersion ϕ_V.

DEFINITION 3.9. Let (M, g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, V_0 a submanifold of M such that $V_0 \subset M^K$ and it has the property $(*)$ with respect to the H-action, $f : V_0 \times [0,T) \to M^K$ a deformation of V_0 in M^K. If the deformation f has its expansion F and the immersed submanifold $V_t := f_t(V_0)$ also has the property $(*)$ with respect to the H-action for each $t \in [0,T)$, we say that the deformation f *preserves the property* $(*)$ with respect to the H-action.

The following theorem shows that we can restrict the condition of mean curvature flows to a condition on some part of $\mathrm{Im}\,\phi_V$ under some assumptions.

Theorem 3.10 ([23]). *Let (M,g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, V_0 a submanifold of M such that $V_0 \subset M^K$ and V_0 has the property $(*)$ with respect to the H-action. Assume that there exists a deformation $f : V_0 \times [0,T) \to M^K$ of V_0 which has its expansion F satisfying the following conditions:*

(i) (restricted MCF condition) *For any $t \in [0,T)$ and any $p \in V_0$, it holds that*

$$\frac{\partial}{\partial t} F_t(K,p) = \mathcal{H}^t(K,p),$$

(ii) *the deformation f preserves the property $(*)$ with respect to the H-action,*

where $\mathcal{H}^t$ is the mean curvature vector field of the immersion $F_t : (H/K) \times V_0 \to M$ for each $t \in [0,T)$. Then the family of maps $(F_t)_{t\in[0,T)}$ is the mean curvature flow of the immersion ϕ_{V_0}.

In general, the restricted MCF condition in Theorem 3.10 is still a partial differential equation. The next corollary tells us conditions that we can reduce the equation to an ordinary differential equation.

Corollary 3.11 ([23]). *Let (M,g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, V_0 a submanifold of M such that $V_0 \subset M^K$ and V_0 has the property $(*)$ with respect to the H-action. Assume that there exists a vector field A along M^K satisfying the following conditions:*

(i.a) *A generates a deformation $f : V_0 \times [0,T) \to M^K$ of V_0 with its expansion F, i.e., it holds that*

$$\frac{\partial}{\partial t} F_t(K,p) = A_{f_t(p)} \quad (p \in V_0,\ t \in [0,T)),$$

(i.b) *it holds that*

$$\mathcal{H}^t(K,p) = A_{f_t(p)} \quad (p \in V_0,\ t \in [0,T)), \text{ and}$$

(ii) *the deformation f preserves the property $(*)$,*

where $\mathcal{H}^t$ is the mean curvature vector field of the immersion $F_t : (H/K) \times V_0 \to M$ for each $t \in [0,T)$. Then, the family of maps $(F_t)_{t\in[0,T)}$ is the mean curvature flow of the immersion ϕ_{V_0}.

Remark 3.12. For a given submanifold $V_0 \subset M^K$, if we can find a vector field A which satisfies the condition (i.b), then the condition (i.a) is an ordinary differential equation.

At the end of this section, we show that the conditions of self-similar solutions and translating solitons also can be reduced to conditions on some part of $\mathrm{Im}\,\phi_V$ under some assumptions.

PROPOSITION 3.13 ([23]). *Let $(\mathbb{R}^n, g)$ be the Euclidean space equipped with the standard Riemannian metric g, H a Lie group which acts on M preserving g, K a closed subgroup of H, V a submanifold of $\mathbb{R}^n$ such that $V \subset (\mathbb{R}^n)^K$ with the property $(*)$. Suppose that for some constant $\lambda \in \mathbb{R}$, it holds that*

$$\mathcal{H}(K, x) = \lambda \phi_V^{\perp}(K, x) \quad (x \in V),$$

where $\phi_V^{\perp}(hK, x)$ denotes the normal part of the position vector $\phi_V(hK, x) \in \mathbb{R}^n$. Then the immersion ϕ_V satisfies the self-similar condition (2), *i.e., it holds that*

$$\mathcal{H}(hK, x) = \lambda \phi_V^{\perp}(hK, x) \quad ((hK, x) \in (H/K) \times V).$$

PROPOSITION 3.14 ([23]). *Let $(\mathbb{R}^n, g)$ be the Euclidean space as above, H a Lie group which acts on M preserving g, K a closed subgroup of H, V a submanifold of $\mathbb{R}^n$ such that $V \subset (\mathbb{R}^n)^K$ with the property $(*)$. Suppose that for some constant vector $v \in \mathbb{R}$, it holds that*

$$\mathcal{H}(K, x) = v^{\perp}(K, x) \quad (x \in V),$$

where $v^{\perp}(hK, x)$ denotes the normal part of the vector v at $\phi_V(hK, x)$. Then the immersion ϕ_V satisfies the translating soliton condition (3), *i.e., it holds that*

$$\mathcal{H}(hK, x) = v^{\perp}(hK, x) \quad ((hK, x) \in (H/K) \times V).$$

4. Lagrangian immersions with symmetries

In this section, we show a way to construct a Lagrangian immersion in a symplectic manifold by a moment map. Moreover, we show that an invariant, oriented Lagrangian immersion in a Calabi-Yau manifold has the Lagrangian angle and the mean curvature vector field with symmetries under some assumptions.

PROPOSITION 4.1 ([23]). *Let (M, ω) be a symplectic manifold, H a Lie group which acts on M preserving ω and has a moment map $\mu : M \to \mathfrak{h}^*$, V_c a sub-*

manifold of M such that $V_c \subset M^K$, $\phi_{V_c} : (H/K) \times V_c \to M; (hK, p) \mapsto hp$ an immersion. Assume that the following conditions hold:

(i) *V_c is isotropic, and*

(ii) (moment map condition) $V_c \subset \mu^{-1}(c)$ for $c \in Z(\mathfrak{h}^*)$.

Then the immersion ϕ_{V_c} is isotropic. Conversely, if the immersion ϕ_{V_c} is isotropic and $\operatorname{Im} \phi_{V_c}$ is connected, then the conditions (i) *and* (ii) *hold.*

Corollary 4.2 ([23]). *In addition to the assumptions of Proposition 4.1, if it holds that*

$$\dim H/K + \dim V_c = n,$$

then the map ϕ_{V_c} is a Lagrangian immersion.

Next we discuss symmetries on Lagrangian angles and mean curvature vector fields which some Lagrangian immersions in Calabi-Yau manifolds have. The overview is as follows. For a given pair of a Calabi-Yau manifold (M, I, ω, Ω) and a Lie group H which acts on M, we define an element a_H of the Lie coalgebra $\mathfrak{h}^*$ of H. Especially, it is shown that a_H is in the center $Z(\mathfrak{h}^*)$ of $\mathfrak{h}^*$. We consider $a_H : \mathfrak{h} \to \mathbb{R}$ to be the the map which expresses rotations of the Calabi-Yau structure Ω for its transformations by the action of H. The covector a_H yields some vector field $I[\alpha_H(\cdot)]^\#$ on some submanifold $V \subset M$. Under some assumptions, the Lagrangian angle and the mean curvature vector of the Lagrangian immersion ϕ_V are expressed by a_H and $I[\alpha_H(\cdot)]^\#$ respectively.

For this purpose, we begin with some fundamental facts as follows.

Let M be a manifold, H a Lie group which acts on M with the Lie algebra $\mathfrak{h}$, K a closed subgroup of H with the Lie subalgebra $\mathfrak{k} \subset \mathfrak{h}$, p a point of M^K. Then the following two maps are well-defined and linearly isomorphic respectively:

$$\mathfrak{h}/\mathfrak{k} \to \mathrm{T}_K(H/K); \quad [\xi] \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)K, \tag{4}$$

$$\mathfrak{h}/\mathfrak{k} \to \mathrm{T}_p(H \cdot p); \quad [\xi] \mapsto \xi_p^\#. \tag{5}$$

So if we define a symbol $[\xi]_p^\#$ by a map $\mathfrak{h}/\mathfrak{k} \to \mathrm{T}_p(H \cdot p); [\xi] \mapsto [\xi]_p^\# := \xi_p^\#$, this symbol is well-defined. We can define a symbol $\frac{d}{dt}\big|_{t=0} \exp(t[\xi])K$ by a map $\mathfrak{h}/\mathfrak{k} \to \mathrm{T}_K(H/K); [\xi] \mapsto \frac{d}{dt}\big|_{t=0} \exp(t[\xi])K := \frac{d}{dt}\big|_{t=0} \exp(t\xi)K$ as well.

With the use of these facts, we can show the following proposition.

Proposition 4.3 ([23]). *Let (M, g) be a Riemannian manifold, H a Lie group which acts on M, K a closed subgroup of H, b an element of $\mathfrak{h}^*$ such that $\mathfrak{k} \subset \operatorname{Ker} b$. Then we can define a map $[\beta(\cdot)] : M^K \to \mathfrak{h}/\mathfrak{k}$; $p \mapsto [\beta(p)]$ by*

$$\langle b, [\eta] \rangle = g_p\big([\beta(p)]^{\#}_p, \eta^{\#}_p\big). \tag{6}$$

The condition that $\mathfrak{k} \subset \operatorname{Ker} b$ in Proposition 4.3 is used for considering $b \in \mathfrak{h}^*$ to be in $(\mathfrak{h}/\mathfrak{k})^*$.

Definition 4.4. Under the conditions of Proposition 4.3, we call the vector field $[\beta(\cdot)]^{\#}$ along M^K defined by Proposition 4.3 *the vector field generated by a covector $b \in \mathfrak{h}^*$ with respect to g.*

In addition, when there exists an almost complex structure I on M, we can define a vector field $I[\beta(\cdot)]^{\#}$ along M^K and call it *the vector field generated by a covector $b \in \mathfrak{h}^*$ with respect to (g, I).*

Next proposition shows a property which the vector field $I[\beta(\cdot)]^{\#}$ has.

Proposition 4.5 ([23]). *Let (M, I, ω, g) be a Kähler manifold, H a Lie group which acts on M with the Lie algebra $\mathfrak{h}$ and is Hamiltonian, i.e., it has a moment map $\mu : M \to \mathfrak{h}^*$, K a closed subgroup of H with the Lie subalgebra $\mathfrak{k} \subset \mathfrak{h}$, b an element of $\mathfrak{h}^*$ such that $\mathfrak{k} \subset \operatorname{Ker} b$. Then, for any $p \in M^K$, it holds that*

$$(d\mu)_p I_p [\beta(p)]^{\#}_p = -b.$$

Corollary 4.6 ([23]). *Suppose the setting of Proposition 4.5. Let $p \in M^K$, $c_0 := \mu(p) \in \mathfrak{h}^*$. Assume that there exists the integral curve $\gamma_p : [0, T) \to M^K$ generated by the vector field $I[\beta(\cdot)]^{\#}$ with the initial condition $\gamma(0) = p$. Then it holds that*

$$\mu(\gamma_p(t)) = c_t, \quad c_t := c_0 - tb.$$

Next we show a formula on transformations of the Calabi-Yau structure by actions of Lie groups and define an element $a_H \in Z(\mathfrak{h}^*)$.

Proposition 4.7 ([22],[23]). *Let (M, I, ω, Ω) be a connected Calabi-Yau manifold and H a connected Lie group which acts on M preserving I and ω. Then there exists $a_H \in Z(\mathfrak{h}^*)$ such that for any $h \in H$, it holds that*

$$L_h^* \Omega = e^{\sqrt{-1}\langle a_H, \eta_1 + \cdots + \eta_l \rangle} \Omega,$$

where

$$\eta_1, \cdots, \eta_l \in \mathfrak{h} \text{ such that } h = \exp\eta_1 \cdots \exp\eta_l.$$

The next proposition shows that we can consider the element a_H to be the differential map of a Lagrangian angle under some conditions.

Proposition 4.8 ([22],[23]). *Let (M, I, ω, Ω) be a $2n$-dimensional connected Calabi-Yau manifold, H a connected Lie group which acts on M preserving I and ω and has a moment map $\mu : M \to \mathfrak{h}^*$, K a closed subgroup of H such that H/K is orientable, V_c an orientable submanifold of M such that $V_c \subset M^K$ satisfying the following conditions.*

(i) *ϕ_{V_c} is an immersion,*

(ii) *V_c is isotropic,*

(iii) (moment map condition) *$V_c \subset \mu^{-1}(c)$ for $c \in Z(\mathfrak{h}^*)$, and*

(iv) *$\dim(H/K) + \dim V_c = n$.*

Then the map ϕ_{V_c} is a Lagrangian immersion by Corollary 4.2. Let θ_c be the Lagrangian angle of ϕ_{V_c}. Then it holds that

$$\theta_c(hK, p) = \theta_c(K, p) + \langle a_H, \eta_1 + \cdots + \eta_l\rangle \quad (hK \in H/K,\ p \in V_c)$$

where a_H is the element of $Z(\mathfrak{h}^*)$ defined by Proposition 4.7, and $h = \exp\eta_1 \cdots \exp\eta_l$.

For any $k \in K$, we have

$$\theta_c(K, p) = \theta_c(kK, p) = \theta_c(K, p) + \langle a_H, \kappa_1 + \cdots + \kappa_l\rangle$$

by Proposition 4.8, where $k = \exp\kappa_1 \cdots \exp\kappa_l$. So we have the following corollary.

Corollary 4.9 ([22],[23]). *Under the conditions of Proposition 4.8, the K-action preserves the Calabi-Yau structure Ω, i.e., it holds that*

$$L_k^*\Omega = \Omega \quad (k \in K).$$

Since we have $\mathfrak{k} \subset \ker a_H$ by Corollary 4.9, we can consider the element $a_H \in \mathfrak{h}^*$ to be one in $(\mathfrak{h}/\mathfrak{k})^*$. Then we can define the vector field $I[\alpha_H(\cdot)]^{\#}$ generated by a_H along $V_c \subset M^K$ with respect to (g, I). At the end of this section, we show that the mean curvature vector fields of some Lagrangian immersions are expressed by $I[\alpha_H(\cdot)]^{\#}_{\cdot}$.

PROPOSITION 4.10 ([23]). *Under the conditions of Proposition 4.8, for any $(hK,p) \in (H/K) \times V_c$, it holds that*

$$\mathcal{H}^c(hK,p) = (L_h)_{*p} I_p \Big\{ [\alpha_H(p)]^{\#}_p + \Big(\mathrm{grad}_{\phi_{V_c}{}^* g} \theta_c(K, \cdot\,) \Big)_p \Big\},$$

where $\mathcal{H}^c$ is the mean curvature vector field of the immersion ϕ_{V_c}.

COROLLARY 4.11 ([23]). *In addition to the conditions of Proposition 4.8, assume that the Lagrangian angle θ_c is constant on $\{K\} \times V_c$. Then it holds that*

$$\mathcal{H}^c(hK,p) = (L_h)_{*p} I_p [\alpha_H(p)]^{\#}_p \quad ((hK,p) \in (H/K) \times V_c).$$

5. Constructions of Lagrangian mean curvature flows

In this section, we apply Corollary 3.11 and Corollary 4.11 to construct Lagrangian mean curvature flows in Calabi-Yau manifolds by symmetries of Lie groups which are not necessarily abelian. In particular, we construct Lagrangian mean curvature flows with the use of generalized perpendicular symmetries. As special cases, we also have methods of constructing special Lagrangian submanifolds.

THEOREM 5.1 ([23]). *Let (M, I, ω, Ω) be a $2n$-dimensional connected Calabi-Yau manifold, H a connected Lie group with the Lie algebra $\mathfrak{h}$ which acts on M preserving I and ω and has a moment map $\mu : M \to \mathfrak{h}^*$, K a closed subgroup of H with the Lie algebra $\mathfrak{k}$ such that H/K is orientable, $m := \dim(H/K)$, a_H the element of $Z(\mathfrak{h}^*)$ defined by Proposition 4.7, $I[\alpha_H(\cdot)]^{\#}_{\cdot}$ the vector field along M^K generated by the covector a_H with respect to (I, g), V_{c_0} an $(n-m)$-dimensional orientable submanifold of M in $M^K \cap \mu^{-1}(c_0)$ for $c_0 \in Z(\mathfrak{h}^*)$ such that the map $\phi_{V_{c_0}}$ is an immersion.*

Suppose that the vector field $I[\alpha_H(\cdot)]^{\#}_{\cdot}$ generates the deformation $f : V_{c_0} \times [0,T) \to M^K$ of V_{c_0} with its expansion F, i.e., the formula

$$\frac{\partial}{\partial t} F_t(K,p) = I_{f_t(p)} [\alpha_H(f_t(p))]^{\#}_{f_t(p)} \quad (p \in V_{c_0}, t \in [0,T)) \tag{7}$$

holds and that for each $t \in [0,T)$ and $c_t := c_0 - t a_H \in Z(\mathfrak{h}^*)$, the following conditions hold:

(i) *the submanifold $V_{c_t} := f_t(V_{c_0})$ is isotropic and*

(ii) *the Lagrangian angle* θ_{c_t} *of the immersion* $\phi_{V_{c_t}}$ *is constant on* $\{K\} \times V_{c_t}$.

Then the family of maps $(F_t)_{t\in[0,T)}$ *is the Lagrangian mean curvature flow of the immersion* $\phi_{V_{c_0}}$, *i.e., it holds that*

$$\frac{\partial}{\partial t} F_t(hK, p) = \mathcal{H}^t(hK, p) \quad \big((hK, p) \in (H/K) \times V_{c_0},\ t \in [0, T)\big).$$

Note that the notation V_{c_t} in Theorem 5.1 is appropriate by Corollary 4.6 and that each $\phi_{V_{c_t}}$ is a Lagrangian immersion by Corollary 4.2.

Next we give more concrete conditions which realize the conditions of Theorem 5.1. For this purpose, we use another given Lagrangian submanifold L and Lie group actions which are perpendicular to L. The following proposition shows the conditions that the Lagrangian angle of the immersion ϕ_{V_c} is expressed by the covector $a_H \in Z(\mathfrak{h}^*)$ and the Lagrangian angle of L.

Proposition 5.2 ([22],[23]). *Let* $(M, I, \omega, g, \Omega)$ *be a connected* $2n$-*dimensional Calabi-Yau manifold and* H *a connected Lie group with the Lie algebra* $\mathfrak{h}$ *which acts on* M *preserving* I *and has a moment map* $\mu : M \to \mathfrak{h}^*$, K *a closed subgroup of* H *with the Lie algebra* $\mathfrak{k}$ *such that* H/K *is orientable,* $m := \dim(H/K)$, L *an oriented Lagrangian submanifold of* M *with its Lagrangian angle* θ, V_c *an* $(n-m)$-*dimensional submanifold of* M *in* L^K. *Assume the following conditions:*

(i) (generalized perpendicular condition) *For any* $p \in V_c$ *and any* $\xi \in \mathfrak{h}$, *the followings hold:*

(i.a) $\xi^\#_p \in \mathrm{T}^\perp_p L \oplus \mathrm{T}_p V_c$,

(i.b) $\xi^\#_p \notin \mathrm{T}_p V_c \backslash \{0\}$, *and*

(ii) (moment map condition) $V_c \subset \mu^{-1}(c)$ *for* $c \in Z(\mathfrak{h}^*)$.

Then V_c *has a canonical orientation. Moreover, the Lagrangian angle* θ_c *of* ϕ_{V_c} *satisfies the following formula:*

$$\theta_c(hK, p) = \theta(p) - \frac{\pi}{2} m + \langle a_H, \eta_1 + \cdots + \eta_l \rangle, \tag{8}$$

where $h = \exp \eta_1 \cdots \exp \eta_l$ *and* a_H *is the element of* $Z(\mathfrak{h}^*)$ *defined by Proposition 4.7.*

We show the conditions that realize the conditions of Theorem 5.1 with the use of Proposition 5.2.

THEOREM 5.3 ([23]). *Let (M, I, ω, Ω) be a $2n$-dimensional connected Calabi-Yau manifold, H a connected Lie group with the Lie algebra $\mathfrak{h}$ which acts on M preserving I and ω and has a moment map $\mu : M \to \mathfrak{h}^*$, K a closed subgroup of H with the Lie algebra $\mathfrak{k}$, $m := \dim(H/K)$, a_H the element of $Z(\mathfrak{h}^*)$ defined by Proposition 4.7, $I[\alpha_H(\cdot)]^{\#}$ the vector field along M^K generated by the covector a_H with respect to (I, g), L an oriented Lagrangian submanifold of M with its Lagrangian angle θ, V_{c_0} an $(n-m)$-dimensional submanifold of M in $L^K \cap \mu^{-1}(c_0)$ for $c_0 \in Z(\mathfrak{h}^*)$ such that the map $\phi_{V_{c_0}}$ is an immersion.*

Suppose that the vector field $I[\alpha_H(\cdot)]^{\#}$ generates the deformation $f : V_{c_0} \times [0, T) \to L^K$ of V_{c_0} with its expansion F, i.e., the formula (7) holds and that for each $t \in [0, T)$, the following conditions hold:

(i) *the generalized perpendicular condition in Proposition 5.2 holds and*

(ii) *the Lagrangian angle θ is constant on V_{c_t} (e.g. L is a special Lagrangian submanifold).*

Then the family of maps $(F_t)_{t\in[0,T)}$ is the Lagrangian mean curvature flow of $\phi_{V_{c_0}}$.

COROLLARY 5.4 ([22],[23]). *In addition to the settings in Theorem 5.3, if the H-action preserves the Calabi-Yau structure Ω, i.e., it holds that $a_H = 0$, then the family of maps (F_t) is the stationary solution of the Lagrangian mean curvature flow of the immersion $\phi_{V_{c_0}}$. That is, $F_t \equiv F_0$ holds for any t and F_0 is a special Lagrangian immersion.*

6. Examples of special Lagrangian submanifolds

In this section, with the use of generalized perpendicular symmetries as shown in Corollary 5.4, we construct special Lagrangian submanifolds in the cotangent bundle T^*S^n of the n-sphere S^n which equipped with the Stenzel metric. Before that, we review some fundamental facts about the Stenzel metrics.

In [24], Stenzel constructed complete Ricci-flat Kähler metrics on the cotangent bundles of compact rank one symmetric spaces, for example on the cotangent bundle T^*S^n of the n-sphere S^n. We denote this Calabi-Yau structure on T^*S^n by $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}})$. We identify the tangent bundle and the cotangent bundle of S^n as follows.

$$\mathrm{T}^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1,\ x \cdot \xi = 0\},$$

where " $\cdot$ " is the canonical real inner product on the Euclidean space $\mathbb{R}^{n+1}$ and $\|x\| := \sqrt{x \cdot x}$ for each $x \in \mathbb{R}^{n+1}$. A cohomogeneity one $SO(n+1)$-action on T^*S^n

is defined by $h \cdot (x, \xi) = (hx, (L_h)_{*x}\xi)$ for $h \in SO(n+1)$. The principal orbit at a point (x, ξ) is a sphere bundle with a radius of $\|\xi\|$.

Let Q^n be a complex quadric hypersurface in $\mathbb{C}^{n+1}$ defined by

$$Q^n = \left\{ z = {}^t(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

In [26], Szöke shows that the following map Φ is an $SO(n+1)$-equivariant diffeomorphism from T^*S^n to Q^n.

$$\begin{aligned} \Phi : \mathrm{T}^*S^n &\to Q^n \\ \cup\!\!\!\!\!\!\;\;\; & \quad \cup\!\!\!\!\!\!\;\;\; \\ (x, \xi) &\mapsto \cosh(\|\xi\|)x + \sqrt{-1}\frac{\sinh(\|\xi\|)}{\|\xi\|}\xi. \end{aligned}$$

We can induce a complex structure I to Q^n from $\mathbb{C}^{n+1}$ by the map Φ. The Kähler form ω_{Stz} is given by the following.

$$\omega_{\mathrm{Stz}} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=1}^{n+1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(r^2) dz_i \wedge d\bar{z}_j,$$

where $r^2 = \|z\|^2 = \sum_{i=1}^{n+1} z_i \bar{z}_i$ and u is a smooth real function which satisfies the following ordinary differential equation:

$$\frac{d}{dt}(U'(t))^n = cn(\sinh t)^{n-1} \quad (c = \text{const.} > 0), \tag{9}$$

where $U(t) = u(\cosh t)$. The functions U and u satisfy $U'(t) > 0, U''(t) > 0$ and $u'(t) > 0$ if $t > 0$ under appropriate choices of a constant of integration (See [24]).

We can verify that the $SO(n+1)$-action preserves the Calabi-Yau structure $(\mathrm{T}^*S^n, I, \omega_{\mathrm{Stz}}, \Omega_{\mathrm{Stz}})$. So we see that the covector a_H in $Z(\mathfrak{h}^*)$ equals 0.

In [2], Anciaux gave a moment map $\mu : Q^n \to \mathfrak{so}(n+1)^*$ with respect to the $SO(n+1)$-action which is defined by the following.

$$(\mu(z))(X) = u'(r^2) Iz \cdot Xz, \quad (z \in Q^n, X \in \mathfrak{so}(n+1)).$$

We use the following fact which was shown by Karigiannis–Min-Oo in [18] for preparing an original special Lagrangian submanifold L. That is, a conormal bundle $\mathrm{T}^{*\perp}N$ in T^*S^n for a submanifold N in S^n is a special Lagrangian if and only if N is austere. In particular, a totally geodesic submanifold of S^n is austere.

6.1. The case of $H = U(1), L_1 = \mathrm{T}^{*\perp} S^2, L_2 = \mathrm{T}^{*\perp} S^1 \subset \mathrm{T}^* S^5$

In this subsection, we construct special Lagrangian submanifolds, using abelian $U(1)$-actions. We show two cases which include a case of strictly perpendicular symmetries and a case of generalized perpendicular symmetries.

Let M be the cotangent bundle of 5-sphere $\mathrm{T}^* S^5$, L_1 the conormal bundle of a submanifold $S^2 \subset S^5$, and L_2 the conormal bundle of a submanifold $S^1 \subset S^5$ defined by the following.

$$L_1(\cong \mathrm{T}^{*\perp} S^2) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \end{bmatrix} \right) \middle| \ \|x\| = 1, \ \xi_j \in \mathbb{R} \ (j = 2, 4, 6) \right\},$$

$$L_2(\cong \mathrm{T}^{*\perp} S^1) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} \right) \middle| \ \|x\| = 1, \ \xi_j \in \mathbb{R} \ (j = 2, 4, 5, 6) \right\}.$$

L_1 and L_2 are special Lagrangian since S^2 and S^1 are totally geodesic in S^5 respectively.

Let H be a Lie group $U(1)$ and define a diagonal $U(1)$-action by the following.

$$SO(2) \times \mathrm{T}^* S^5 \ni (h, (x, \xi)) \mapsto \left[\begin{array}{c|c|c} h & & \\ \hline & h & \\ \hline & & h \end{array} \right] \cdot (x, \xi) \in \mathrm{T}^* S^5.$$

Here, we identify $U(1)$ with $SO(2)$. That is, this action is the Hopf-fibration $S^5 \to \mathbb{C}P^2$. The isotropy subgroup of this $U(1)$-action is trivial at any point $p \in L_i$ for $i = 1, 2$. The moment map μ given by [2] is calculated as follows.

$$\begin{cases} -\mathcal{K}(\|\xi\|)(x_1\xi_2 + x_3\xi_4 + x_5\xi_6) & \text{on } \Phi(L_1) \backslash \{\|\xi\| = 0\}, \\ -\mathcal{K}(\|\xi\|)(x_1\xi_2 + x_3\xi_4) & \text{on } \Phi(L_2) \backslash \{\|\xi\| = 0\}, \end{cases}$$

where,

$$\mathcal{K}(\|\xi\|) = \frac{u'(\cosh(2\|\xi\|))\sinh(2\|\xi\|)}{\|\xi\|}.$$

Let $V_c^{(i)} := L_i \cap \mu^{-1}(c)$ $(i = 1, 2)$ for $c \in \mathfrak{h}^*$. We can verify that the $U(1)$-action satisfies the strictly perpendicular condition on $V_c^{(1)}$ to L_1 and the generalized perpendicular condition on $V_c^{(2)}$ to L_2 by direct calculations. Thus we have the following result.

Proposition 6.1 ([22]). *For $i = 1, 2$, $H \cdot V_c^{(i)}$ is a special Lagrangian submanifold for any $c \in \mathfrak{h}^*$ if $V_c^{(i)}$ is not the empty set.*

The proof is based on Corollary 5.4 (See [22]).

6.2. The case of $H = SO(2) \times SO(2) \times SO(3), L = \mathrm{T}^{*\perp} S^2 \subset \mathrm{T}^* S^6$

In this subsection, we construct special Lagrangian submanifolds, using non-abelian H-actions. In this case, we use the strictly perpendicular symmetries.

Let M be the cotangent bundle of 6-sphere T^*S^6, L the conormal bundle of a totally geodesic submanifold $S^2 \subset S^6$ defined by the following.

$$L(\cong \mathrm{T}^{*\perp} S^2) = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ \xi_6 \\ \xi_7 \end{bmatrix} \right) \;\middle|\; \|x\| = 1,\ \xi_j \in \mathbb{R}\ (j = 2, 4, 6, 7) \right\}.$$

Let H be a Lie group $SO(2) \times SO(2) \times SO(3)$ and define an H-action by the following.

$$H \times \mathrm{T}^*S^6 \ni \big(h_1, h_2, h_3, (x, \xi)\big) \mapsto \left[\begin{array}{c|c|c} h_1 & 0 & 0 \\ \hline 0 & h_2 & 0 \\ \hline 0 & 0 & h_3 \end{array}\right] \cdot (x, \xi) \in \mathrm{T}^*S^6,$$

where, $h_1, h_2 \in SO(2)$ and $h_3 \in SO(3)$. Note that H is non-abelian and the center of the Lie coalgebra $\mathfrak{h}^*$ is given by $Z(\mathfrak{h}^*) \cong \mathfrak{so}(2)^* \oplus \mathfrak{so}(2)^*$. This indicates that we can expect to obtain two-parameter family of special Lagrangian submanifolds by our method.

The moment map μ given by [2] is calculated as follows. Let ξ_{ij} be the $(n+1)\times$

$(n+1)$-matrix defined by $\xi_{ij} = E_{ji} - E_{ij}$, where E_{ij} denotes the $(n+1)\times(n+1)$-matrix whose (i,j)-component is 1 and all the others are 0. Then we see that

$$\mathfrak{h}(\cong \mathfrak{so}(2)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(3)) = \mathrm{span}\{\xi_{12},\xi_{34},\xi_{56},\xi_{57},\xi_{67}\}.$$

For each $(i,j) = (1,2),(3,4),(5,6),(5,7),(6,7)$, define a map $\mu_{ij} : Q^6 \to \mathbb{R}$ by $\mu_{ij}(z) = \langle \mu(z), \xi_{ij}\rangle$. Then by direct calculations, we have

$$\begin{aligned}
&\mu_{12}(z) = -\mathcal{K}(\|\xi\|)x_1\xi_2,\\
&\mu_{34}(z) = -\mathcal{K}(\|\xi\|)x_3\xi_4,\\
&\mu_{56}(z) = -\mathcal{K}(\|\xi\|)x_5\xi_6,\\
&\mu_{57}(z) = -\mathcal{K}(\|\xi\|)x_5\xi_7,\\
&\mu_{67}(z) \equiv 0
\end{aligned}$$

on $\Phi(L)\backslash\{\|\xi\|=0\}$, where

$$\mathcal{K}(\|\xi\|) = \frac{u'(\cosh(2\|\xi\|))\sinh(2\|\xi\|)}{\|\xi\|}.$$

We set the following rank two subbundle $\hat{L}\subset L$ so that the moment map condition is satisfied on $\hat{L}$.

$$\hat{L} = \left\{ \left(\begin{bmatrix} x_1\\0\\x_3\\0\\x_5\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\ \xi_2\\0\\ \xi_4\\0\\0\\0 \end{bmatrix} \right) \,\middle|\, \|x\|=1,\ \xi_j\in\mathbb{R}\ (j=2,4) \right\}.$$

Let K be a closed subgroup of H defined by

$$K(\cong SO(2)) = \left\{ \left[\begin{array}{c|c} E_5 & \\ \hline & h \end{array}\right] \,\middle|\, h\in SO(2) \right\},$$

where E_5 is the unit 5×5-matrix. We see that the isotropy subgroup at a generic point $p\in\hat{L}$ equals K.

For $(c_1, c_2) \in \mathbb{R}^2$, define $V_{(c_1,c_2)}$ and $W_{(c_1,c_2)}$ by the following.

$$V_{(c_1,c_2)} = \hat{L}^K \cap \{p \in M \mid \mu_{12}(p) = c_1,\ \mu_{34}(p) = c_2,\ \mu_{ij}(p) = 0\},$$
$$W_{(c_1,c_2)} = \hat{L} \cap \{p \in M \mid \mu_{12}(p) = c_1,\ \mu_{34}(p) = c_2,\ \mu_{ij}(p) = 0\ \},$$

where $(i,j) = (5,6), (5,7), (6,7)$. We can verify that the H-action satisfies the strictly perpendicular condition on $V_{(c_1,c_2)}$ to L by direct calculations. Thus we have the following result.

PROPOSITION 6.2 ([22]). *For any $(c_1, c_2) \neq (0,0) \in \mathbb{R}^2$ such that $V_{(c_1,c_2)}$ is not the empty set, $H \cdot V_{(c_1,c_2)}$ is a special Lagrangian submanifold, and $H \cdot W_{(0,0)}$ is a union of five connected special Lagrangian submanifolds.*

The former of this claim is shown by Corollary 5.4.

$W_{(0,0)}$ includes non-principal points and is not a smooth manifold. We can verify that it is a union, which is not disjoint, of the following five connected submanifolds.

$$W_{(0,0)} = W^{S^2}_{(0,0)} \cup W^{S^1\times\mathbb{R}}_{(0,0),(1)} \cup W^{S^1\times\mathbb{R}}_{(0,0),(3)} \cup W^{\mathbb{R}^2}_{(0,0),(1)} \cup W^{\mathbb{R}^2}_{(0,0),(-1)},$$

where

$$W^{S^2}_{(0,0)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{0} \right) \middle| \ \|x\| = 1 \right\}, \quad W^{S^1\times\mathbb{R}}_{(0,0),(1)} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \|x\| = 1, \\ \xi_2 \in \mathbb{R} \end{array} \right\},$$

$$W^{S^1\times\mathbb{R}}_{(0,0),(3)} = \left\{ \left(\begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ x_5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \|x\| = 1, \\ \xi_4 \in \mathbb{R} \end{array} \right\}, \quad W^{\mathbb{R}^2}_{(0,0),(\epsilon)} = \left\{ \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \xi_2 \\ 0 \\ \xi_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \middle| \begin{array}{c} \xi_i \in \mathbb{R}, \\ i = 2, 4 \end{array} \right\},$$

and $\epsilon = \pm 1$. We see that each set $W^A_{(0,0)}$ above is a 2-dimensional connected submanifold of T^*S^6 diffeomorphic to A. We can not apply our method to each $W^A_{(0,0)}$, since they have non-principal points. However we can directly verify that each $H \cdot W^A_{(0,0)}$ for $W^{S^2}_{(0,0)}$, $W^{S^1\times\mathbb{R}}_{(0,0),(j)}(j=1,3)$, and $W^{\mathbb{R}^2}_{(0,0),(\epsilon)}(\epsilon=\pm 1)$ is a special Lagrangian submanifold of T^*S^n diffeomorphic to S^6, $\mathrm{T}^{*\perp}S^4$, and $\mathrm{T}^{*\perp}S^2$ respectively.

Thus we obtain two-parameter family of special Lagrangian submanifolds in T^*S^6. We can verify that H acts on $H \cdot V_{(c_1,c_2)}$ with cohomogeneity two.

7. Examples of Lagrangian mean curvature flows

In this section, we construct a self-similar solution of a Lagrangian mean curvature flow in $\mathbb{C}^n$ by Theorem 5.3. We use strictly (not generalized) perpendicular symmetries of a non-abelian Lie group $U(1) \times SO(3)$.

Let $(\mathbb{C}^4, I, \omega, \Omega, g)$ be a 4-dimensional complex Euclidean space equipped with the standard Calabi-Yau structure, where I, ω, Ω and g are the complex structure, the Kähler form, the Calabi-Yau structure, and the Kähler metric respectively. Let L be a special Lagrangian submanifold defined by

$$L := \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \mathbf{0} \right) \in \mathbb{R}^4 \times \mathbb{R}^4 \;\middle|\; x_i \in \mathbb{R}\ (i=1,2,3,4) \right\} \cong \mathbb{R}^4.$$

Let H be a Lie group $U(1) \times SO(3)$. For $\lambda_1, \lambda_2 \in \mathbb{Z}$, we define an action of H to $\mathbb{C}^4$ by

$$H \times \mathbb{C}^4 \ni (e^{\sqrt{-1}\theta}, h) \cdot z := \left[\begin{array}{c|c} 1 & \\ \hline & h \end{array}\right] \left[\begin{array}{c|c} e^{\sqrt{-1}\lambda_1\theta} & \\ \hline & e^{\sqrt{-1}\lambda_2\theta} E_3 \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \in \mathbb{C}^4.$$

We can verify this action preserves I and ω by direct computations.

Let K be a closed subgroup of H defined by

$$K := \left\{ \left[\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & k \end{array}\right] \;\middle|\; k \in SO(2) \right\} \cong SO(2).$$

Then we have

$$L^K = \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{0} \right) \in \mathbb{R}^4 \times \mathbb{R}^4 \;\middle|\; x_2 \neq 0 \right\}.$$

We can verify that the action of H is strictly perpendicular to L on L^K by direct computations.

Next we consider a moment map μ. Let $\mathfrak{h}$ be the Lie algebra of H. The coalgebra $\mathfrak{h}^*$ is $\mathfrak{u}(1)^* \oplus \mathfrak{so}(3)^*$. The center $Z(\mathfrak{h}^*)$ of $\mathfrak{h}^*$ is $\mathfrak{u}(1)^*$. Define $\xi_1 \in \mathfrak{h}$ by

$$\exp(t\xi_1) \cdot z = \left[\begin{array}{c|c} 1 & \\ \hline & E_3 \end{array} \right] \left[\begin{array}{c|c} e^{\sqrt{-1}\lambda_1 t} & \\ \hline & e^{\sqrt{-1}\lambda_2 t} E_3 \end{array} \right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}.$$

Then by direct calculations, we have

$$\langle \mu(z), \xi_1 \rangle = \frac{1}{2}(\lambda_1 |z_1|^2 + \lambda_2 |z_2|^2 + \lambda_2 |z_3|^2 + \lambda_2 |z_4|^2).$$

So we define a submanifold V_c by

$$V_c := L^K \cap \mu^{-1}(c\xi^1) = \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{0} \right) \in L^K \;\middle|\; \frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2) = c \right\},$$

where $\xi^1 \in \mathfrak{h}^*$ is the dual element of ξ_1 and $c \in \mathbb{R}$. We see that $\dim V_c + \dim(H/K) = 4 = (\dim_{\mathbb{R}} \mathbb{C}^4)/2$. Assume $c \neq 0$. Then we see that V_c is an ellipse if $\lambda_1 > 0$ and $\lambda_2 > 0$, that it is a hyperbola if $\lambda_1 \lambda_2 < 0$ and that V_c is the empty set if $\lambda_1 < 0$ and $\lambda_2 < 0$.

We can calculate that $a_H = (\lambda_1 + 3\lambda_2)\xi^1$. From this formula, we also see that the vector field $I[\alpha_H(\cdot)]^\#$ is given by

$$I_z[\alpha_H(z)]_z^\# = -\frac{\lambda_1 + 3\lambda_2}{\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2} \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ 0 \\ 0 \end{bmatrix} \in \mathrm{T}_z L^K \quad \bigl(z \in L^K\bigr).$$

PROPOSITION 7.1 ([23]). *Fix $c_0 \neq 0$. Then the map $\phi_{V_{c_0}} : \Big((U(1) \times SO(3))/SO(2)\Big) \times V_{c_0} \to \mathbb{C}^4$ generates a Lagrangian mean curvature flow (F_t). Moreover, this flow (F_t) is a self-similar solution of the mean curvature flow. That is, for $k_{c_0} := -(\lambda_1 + 3\lambda_2)/c_0$, the flow F_t is a self-shrinker if $k_{c_0} < 0$ and is a self-expander if $k_{c_0} > 0$.*

The former of this claim is based on Theorem 5.3. The latter is shown by Proposition 3.13. This example is the generalization of ones by Lee–Wang [20] and Konno [19].

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